Midterm Examination II
"SOLUTIONS"

Question #1: ________________/30
Question #2: ________________/30
Question #3: ________________/40

Grand Total $\sum$: ____________________/100

Examination Number #: ___________________
Solutions

Midterm Examination II

Please Place your name of the BACK of the LAST PAGE of the examination!

There are three questions on this examination. The three questions are open-ended with point values as indicated. Please answer all questions and show all work on this examination paper. Please answer all questions and show all work on your examination paper. You may use the back of the sheets if necessary

Question I 30 points:

A company is planning its advertising strategy for next year for its three major products. Since the three products are quite different, each advertising effort will focus on a single product. In units of millions of dollars, a total of 6 is available for advertising next year, where the advertising expenditure for each product must be an integer greater than or equal to 1. The vice-president for marketing has established the objective: Determine how much to spend on each product in order to maximize total sales. The following table gives the estimated increase in sales (in appropriate units) for the different advertising expenditures.

<table>
<thead>
<tr>
<th>Advertising Expenditure</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>14</td>
<td>11</td>
<td>13</td>
</tr>
<tr>
<td>4</td>
<td>17</td>
<td>14</td>
<td>15</td>
</tr>
</tbody>
</table>

Use dynamic programming to solve this problem.

For \( n = 3 \)

\[
\begin{array}{c|c|c}
\text{Advertising Expenditure} & f_3^x(x_3) & x_3^x \\
\hline
1 & 6 & 1 \\
2 & 9 & 2 \\
3 & 13 & 3 \\
4 & 15 & 4 \\
\end{array}
\]
Question I (continued)

For $n = 2$

$$f^*_2(s, x_2) = h_2(s, x_2) + f^*_2(s, x_2 - x_2)$$

<table>
<thead>
<tr>
<th>$x_2$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>$f^*_2(s)$</th>
<th>$x_2^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
<td>10</td>
<td></td>
<td></td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>14</td>
<td></td>
<td></td>
<td>14</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>17</td>
<td>17</td>
<td>17</td>
<td></td>
<td>17</td>
<td>1, 2, 3</td>
</tr>
<tr>
<td>5</td>
<td>19</td>
<td>21</td>
<td>20</td>
<td>20</td>
<td>21</td>
<td>2</td>
</tr>
</tbody>
</table>

For $n = 1$

$$f^*_1(s, x_1) = h_1(s) + f^*_1(s, x_1 - x_1)$$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>$f^*_1(s)$</th>
<th>$x_1^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>28</td>
<td>27</td>
<td>28</td>
<td>27</td>
<td>28</td>
<td>1, 3</td>
</tr>
</tbody>
</table>

There are two optimal plans

<table>
<thead>
<tr>
<th>Optimal Plan</th>
<th>$x_1^*$</th>
<th>$x_2^*$</th>
<th>$x_3^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>
Question II: 30 points

Customers arrive at a fast food restaurant with one server according to a Poisson process at a mean rate of 30 per hour. The server has just resigned, and the two candidates for the replacement are $X$ (fast but expensive) and $Y$ (slow but inexpensive). Both candidates would have an exponential distribution for service times with $X$ having a mean of 1.2 minutes and $Y$ having a mean of 1.5 minutes. Restaurant revenue per month is given by $\$6,000/W$ where $W$ is the expected waiting time (in minutes) of a customer in the system. Determine the upper bound on the difference in their monthly compensations that would justify hiring $X$ rather than $Y$.

For both alternatives, we have an $M/M/1$ queuing system. Using one hour as the time unit, we have $\lambda = 30 \text{ per hour}$ in both cases.

(10) With candidate $X$, $m = 50$, so:

$$W = \frac{1}{m-\lambda} = \frac{1}{20} = 3 \text{ minutes}$$

Monthly revenue = $\frac{\$6000}{3} = \$2000$.

(10) With candidate $Y$, $m = 40$, so:

$$W = \frac{1}{m-\lambda} = \frac{1}{10} = 6 \text{ minutes}$$

Monthly revenue = $\frac{\$6000}{6} = \$1000$.

(10) Since the difference in monthly revenues is $\$1000$,
The upper bound on the difference in their monthly compensations that would justify hiring candidate $X$ rather than candidate $Y$ is $\$1000$. 

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Question III: 40 points

A small repair shop has been found to have the following characteristics. The shop does not accept any new work when there are 3 jobs in the shop. Of the 2 workers in the shop, the slower worker takes an average of half a day for each repair, and the faster worker takes an average of a third of a day. The same worker who started a job will finish it, and the faster worker always gets called on first. Requests for service arrive at a rate of 4 per day. All distributions are negative exponential.

a) Draw a transition diagram and write down the steady-state balance equations for the states of the system (Hint: you will need at least five states.)

b) Express the equations in a suitable matrix form of the type $A\mathbf{x} = \mathbf{b}$ that could be solved by Gaussian elimination for the limiting probabilities $\mathbf{p}$, but do not attempt to solve.

\[
S : \text{ only slow worker working} \quad (\mu = 2)
\]
\[
F : \text{ " fast " } \quad (\mu = 3)
\]

\[
\begin{align*}
&\text{(25)} \\
&\text{-(15)}
\end{align*}
\]

\[
\begin{align*}
\text{a) rate out} &= \text{ rate in} \\
-(10) \quad 4p_0 &= 2p_5 + 3p_F \\
6p_5 &= 3p_2 \\
7p_F &= 2p_2 + 4p_0 \\
9p_2 &= 5p_3 + 4p_F + 4p_5 \\
p_1 &= p_0 + p_5 + p_F + p_2 + p_3
\end{align*}
\]

\[
\begin{align*}
\text{b) (15)} \\
\begin{bmatrix}
4 & -2 & -3 & 0 & 0 \\
0 & 6 & 0 & -3 & 0 \\
-4 & 0 & 7 & -2 & 0 \\
0 & 4 & -4 & 9 & -5 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
p_0 \\
p_5 \\
p_F \\
p_2 \\
p_3
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
0 \\
1
\end{bmatrix}
\end{align*}
\]