Midterm Examination

Please place your name of the BACK of the LAST PAGE of the examination!

There are three questions on this examination. The first question is a series of 6 (true-false) questions, each worth 5 points each for a total of 30 points. The other two questions are open-ended and point values as indicated. Please answer all questions and show all work on your examination paper. You may use the back of the sheets if necessary.

Question I 30 points:

T (F) a) If X and Y are independent random variables and the moment generating function of each random variable is given as \( \phi_X(t) \), \( \phi_Y(t) \) respectively, then \( \phi_{X+Y} = \phi_X(t) + \phi_Y(t) \).

T (F) b) A Closed Communicating Class (CCC) is a set of states such that if \( i \in C \) and \( j \notin C \), then \( p_{ij} > 0 \).

T (F) c) For the following generating function of a branching problem, \( \psi(u) = \frac{1}{5} u^0 + \frac{1}{2} u^1 + \frac{3}{5} u^2 \) (without actually computing the extinction probability \( u \), one can show that the extinction probability will be \( u < 1 \).

T (F) d) For a finite irreducible DTMC with state space \( S = \{0, 1, \ldots, M\} \), if there is at least one positive element along the diagonal (from \( p_{0,0} \)) of the transition matrix \( P \), then the DTMC is aperiodic.

T (F) e) Consider the DTMC below, then the stationary probability \( \pi_0 = (\alpha/(\alpha + \beta)) \)

\[
P = \begin{pmatrix}
 0 & 1 \\
 1 - \alpha & \alpha & 1 - \beta \\
 \beta & 1 & 1 - \beta
\end{pmatrix}
\]

where \( 0 < \alpha, \beta < 1 \).

\[
\pi_0 = \alpha \pi_0 + (1 - \beta) \pi_1 
\]

\[
\pi_1 = (i-1) \pi_0 + \pi_1 = \frac{\pi_0}{24} b
\]

T (F) f) For the following transition matrix \( P = \begin{pmatrix}
 0 & 1 & 0 \\
 1 & 0 & 1/4 \\
 2 & 1 & 0
\end{pmatrix} \)

the DTMC is reversible. By the Kolmogorov criterion:

\[
p_{0i} = 1/4 \neq p_{0i} = 1/4 \text{ reversible}
\]

\[
p_{2i} = 0
\]

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Question II: 30 points

A wireless router manufacturer is offering a complete replacement warranty if the router fails within the first 3 years. The company has noted that only about 1% of the routers will fail during the first year, whereas 5% of the routers that survive the first year will fail during the second year and 10% of the routers that survived the first two years will fail during the third year. The warranty does not cover the replacement routers.

(a) Model the problem as a DTMC with 5 states and construct the (one-step) probability transition matrix.

(b) Classify the states of the DTMC.

(c) Find the probability that in the first year, the manufacturer will have to honor the warranty!

\[ Q = \begin{bmatrix}
0 & 0.99 & 0 \\
0 & 0 & 0.95 \\
0 & 0 & 0
\end{bmatrix} \]

\[ (I-Q)^{-1} = \begin{bmatrix}
1 & 0.99 & 0.995 \\
0 & 1 & 0.95 \\
0 & 0 & 1
\end{bmatrix} \]

\[ R = \begin{bmatrix}
0.01 & 0 \\
0.05 & 0 \\
0.10 & 0.90
\end{bmatrix} \]

\[ \pi = (I-Q)^{-1} R = \begin{bmatrix} 1.536 & 1.8436 \\ 1.145 & 1.8530 \\ 1.000 & 0.9000 \end{bmatrix} \]

Alternatively: (from the notes:)

\[ u_1 = \sum_{k=1}^{n} \frac{5}{k} \cdot 0.2 \]

\[ u_1 = 0.10 + 0.19 \]

\[ u_2 = 0.25 + 0.95 \]

\[ u_3 = 1.1 \]

\[ u_4 = 1.1 \]

So starting in year one, \( u_1 = 1.536 \)

If honoring the warranty.
Question III: 40 points

The figure below shows a two-stage production system. At each stage, an operation is performed on the part being processed. Parts are introduced into the system at stage 1. Processing is serial, and each stage can hold only one part at a time.

For purposes of analysis, discretize time into 1-minute intervals. At the beginning of an interval, stage 1 is empty, working, or blocked whereas stage 2 is either working or empty. The system operates under the following rules:

- If stage 1 is empty at the beginning of an interval, a new part is introduced into stage 1 with probability 0.9. Work begins on the part; however, it cannot be completed during the minute it is introduced. If stage 1 is working or blocked, no part enters.

- If stage 2 is working at the beginning of an interval, the part will be completed and will leave the system with probability 0.8. Alternatively, it will remain in stage 2 with probability 0.2.

- If stage 1 is working at the beginning of an interval, the part will be completed with probability 0.6. A completed part will move to stage 2 if stage 2 is empty. Otherwise, it is blocked and will remain at stage 1 until stage 2 become empty.

(15) i) Develop a DTMC that describes the production situation.

(10) ii) Draw the corresponding transition network.

(15) iii) Write out the necessary equations to calculate the steady state probabilities, but do not try to solve the equations.

(i) There are five possible states:

- Empty $(0,0)$
- Working $(0,1)$
- Two $(0,1)$
- Both $(1,1)$
- Blocked $(1,0)$

Transition probabilities:

\[
\begin{align*}
&(0,0) & (0,1) & (0,1) & (1,1) & (1,0) \\
(0,0) & 0 & 0 & 0 & 0 & 0 \\
(0,1) & 0 & 0 & 0 & 0 & 0 \\
(0,1) & 0 & 0 & 0 & 0 & 0 \\
(1,1) & 0 & 0 & 0 & 0 & 0 \\
(1,0) & 0 & 0 & 0 & 0 & 0
\end{align*}
\]
One realization of the transition diagram is:

The steady-state equations are derived from the transition matrix:

\[
\begin{align*}
\pi_{00} &= .10 \pi_{00} + \pi_{01} .08 \\
\pi_{10} &= .90 \pi_{00} + .10 \pi_{10} + .72 \pi_{01} + .28 \pi_{11} \\
\pi_{01} &= .60 \pi_{00} + .02 \pi_{01} + .48 \pi_{11} + .52 \pi_{b1} \\
\pi_{11} &= .18 \pi_{01} + .08 \pi_{11} \\
\pi_{b1} &= .12 \pi_{11} + .20 \pi_{b1}
\end{align*}
\]

\[\pi_{00} + \pi_{10} + \pi_{01} + \pi_{11} + \pi_{b1} = 1\]