#1 (10 pts.) A cube has six sides colored red, white, blue, green, yellow, and violet. It is assumed that these six sides are equally likely to show when the cube is tossed. The cube is tossed once.

a) Describe the sample space.

b) Consider the random variable that assigns the number 0 to red and white, the number 1 to green and blue, and the number 2 to yellow and violet. What is the probability distribution of this random variable?

c) Let $Y = (X + 1)^2$, where $X$ is the random variable in part b). Find $E(Y)$.

#2 (10 pts.) Suppose the sample space $S$ consists of the four points:

$$\omega_1, \omega_2, \omega_3, \omega_4$$

and the associated probabilities over the events are given by:

$$P\{\omega_1\} = 1/3, P\{\omega_2\} = 1/5, P\{\omega_3\} = 3/10, P\{\omega_4\} = 1/6$$

Define the random variable $X_1$ by:

$$X_1(\omega_1) = 1, X_1(\omega_2) = 1, X_1(\omega_3) = 4, X_1(\omega_4) = 5$$

and the random variable $X_2$ by:

$$X_2(\omega_1) = 1, X_2(\omega_2) = 1, X_2(\omega_3) = 1, X_2(\omega_4) = 5$$

a) Find the probability distribution of $X_1$, that is $P_{X_1}(i)$.

b) Find $E(X_1)$.

c) Find the probability distribution of the random variable $X_1 + X_2$, that is, $P_{X_1 + X_2}(i)$.

d) Find $E(X_1 + X_2)$ and $E(X_2)$.

#3 (10 pts.) During the course of the day, a machine turns out two items, one in the morning and one in the afternoon. The quality of each item is measured as good ($G$), mediocre ($M$), or bad ($B$). The long-run fraction of good items the machine produces is $1/2$, the fraction of mediocre items is $1/3$, and the fraction of bad items is $1/6$.

a) In a column, write the sample space for the experiment that consists of observing the day’s production.

b) Assume a good item returns a profit of $2$, a mediocre item a profit of $1$, and a bad item yields nothing. Let $X$ be the random variable describing the total profit.
for the day. In a column adjacent to the column in part a), write the value of this random variable corresponding to each point in the sample space.

c) Assuming that the qualities of the morning and afternoon items are independent, in a third column associate with every point in the sample space a probability for that point.

d) Write the set of all possible outcomes for the random variable X. Give the probability distribution function for the random variable.

e) What is the expected value of the day’s profit?

#4 (10 pts.) The life time $X$ in hours, of a certain kind of CRT tube has a probability density function given by:

$$f_X(y) = \begin{cases} \frac{100}{y^2}, & \text{for } y \geq 100 \\ 0, & \text{for } y < 100. \end{cases}$$

a) What is the probability that a tube will survive 250 hours of operation?

b) Find the expected value of the random variable.

#5 (10 pts.) The number of orders per week, $X$, for DVDs can be assumed to have a Poisson distribution with parameter $\lambda = 25$.

a) Find $P\{X \geq 25\}$ and $P\{X = 20\}$.

b) If the number of DVDs in the inventory is 35, what is the probability of a shortage occurring in a week?

#6 (10 pts.) Suppose that a RV $X$ has density $f$ given by:

$$f(x) = \begin{cases} \frac{2}{a^2}(a - x) & \text{for } 0 \leq x \leq a \\ 0 & \text{for } x \geq a \end{cases}$$

where $a$ is a positive constant.

a) Show that the cumulative distribution function $F$ is given by:

$$F(x) = \begin{cases} \frac{2}{a}(2 - \frac{x}{a}) & \text{for } 0 \leq x \leq a \\ 0 & \text{for } x \geq a \end{cases}$$

(b) Show that the mean life time $\mu$ is

$$\mu = \frac{a}{3}$$
and its variance is 
\[ \sigma^2 = \frac{a^2}{18} \]

(c) Sketch the graphs of \( f(x), F(x) \) indicating slopes, etc.

(d) Take \( a = 2 \) hours and compute the pr. that the life time is between 30 and 60 minutes.

#7 (10 pts.) Assume that the RV of a battery in a flashlight has the uniform distribution with a constant density \( 1/L \).

(a) Suppose the mean time was found to be 25 days. Determine \( L \).

(b) Suppose the battery has already lasted 30 days. Find the probability that the flashlight will continue to operate (with the same battery) for more than one week.