"Solutions"
Midterm Examination II

Question #1: _______________/20
Question #2: _______________/25
Question #3: _______________/25
Question #4: _______________/30

Grand Total $\Sigma$: _______________/100

Examination Number #: ___________________
Midterm Examination II

Please Place your name of the BACK of the LAST PAGE of the examination!

There are four questions on this examination. The first question is a series of 5 (true-false) questions, each worth 4 points each for a total of 20 points. The other three questions are open-ended with point values as indicated. Please answer all questions and show all work on this examination paper. Please answer all questions and show all work on your examination paper. You may use the back of the sheets if necessary.

Question I 20 points:

In the following true-false questions, answer all questions, there is no penalty for guessing. Please circle the appropriate response!

◯ F a) In a deterministic dynamic Program (DP), the amount of computations at each stage depends upon the feasible range for the state values.

◯ F b) In a network representation of a DP, the nodes represent the state values at each stage and the arcs represent the feasible alternatives.

◯ (F) c) If the lifetime of a radio is exponentially distributed with a mean value of ten years, the probability it will be working after an additional ten years is most nearly \( \text{Exp}(-10) \).

◯ F d) In an \( M/M/1/\infty \) model with arrival rate \( \lambda \) and service rate \( \mu \), \( \lambda P_0 = \mu P_1 \).

◯ F e) In an \( M/G/\infty \) system model, \( L_q = W_q = 0 \) for such a system.
Question II: 25 points

The manager of a market can hire either Mary or Alice. Mary, who gives service at an exponential rate of 20 customers per hour, can be hired at a hourly rate of $3 per hour. Alice, who gives service at an exponential rate of 30 customers per hour, can be hired at a rate of $C per hour. The manager estimates that, on the average, each customer’s time is worth $1 per hour and should be accounted for in the model. If customers arrive at a Poisson rate of 10 per hour, then

(a) What is the average cost per hour if Mary is hired? If Alice is hired?

b) find C if the average cost per hour is the same for Mary and Alice.

(a) Let $C_M = Mary's$ avg cost/hr and $C_A = Alice's$ cost/hr.

Then $C_M = 3 + 1 = (Avg \ # \ of \ customers \ in \ queue \ when \ Mary \ works.)$

$C_A = C + 1 = " " " " " " Alice " " " ".$

The arrival stream has parameter $\lambda = 10$, and there are two service parameters - one for Mary and one for Alice:

$M_M = 20, \ M_A = 30.$

Set $L_M = \text{avg. # of customers in queue when Mary works and}$

$L_A = " " " " " " Alice " " " "$.

$L_M = \frac{10}{(20-10)} = 1 \quad L_A = \frac{10}{(30-10)} = \frac{1}{2}$

So $C_M = 3 + 1 \times 1 = \frac{4}{hr}.$

$C_A = C + \frac{1}{2} \times \frac{1}{2} = \frac{C + \frac{1}{2}}{hr}.$

(b) If $C_A = C_M$, solve for $C$

$4 = C + \frac{1}{2} \Rightarrow C = \frac{3.50}{hr}.$

i.e., $\frac{3.50}{hr}$ is the most the employer should be willing to pay Alice to work. At a higher cost, his avg. cost is lower with Mary working.
Question III: 25 points

The Won Hung Rhee Chinese carry-out restaurant serves two dishes, chow-mein and spare ribs. There are two separate windows, one for chow-mein and one for spare ribs. Customers arrive according to a Poisson process with a mean rate of 20/hr. Sixty percent go to the chow-mein window while 40% go to the rib window. Twenty percent of those who come from the chow-mein window with their order go next to the rib window; the other 80% leave the restaurant. Ten percent of those who purchase ribs then go to the chow-mein window while the other 90% leave.

It takes on the average 4 minutes to fill a chow-mein order and 5 minutes to fill a spare-rib order, the service times being exponential.

a) How many customers on average are in the restaurant?

b) What is the average wait at each window?

c) If a person wants both chow mein and spare ribs, how long, on the average, does she spend in the restaurant?

\[
\begin{align*}
A &= 20 \\
CM &\rightarrow 180 \rightarrow CM \\
&\quad \downarrow \quad \quad 10 \\
&\quad \downarrow \quad \quad 20 \\
&\quad \downarrow \quad \quad 40 \\
SR &\rightarrow 90 \\
\end{align*}
\]

Solving the traffic equations (1st sys.)

\[
\begin{align*}
\lambda_{CM} &= 12 + 0.10 \lambda_{SR} \\
\lambda_{SR} &= 8 + 0.20 \lambda_{CM}
\end{align*}
\]

\[
\begin{align*}
\lambda_{CM} &= 13.06 \\
\lambda_{SR} &= 10.61
\end{align*}
\]

(2nd system)

\[
\begin{align*}
\lambda_{CM} &\approx 12.8 \\
\lambda_{SR} &\approx 10.4
\end{align*}
\]

Using the network diagram

\[
\begin{array}{c|c|c|c}
\text{Jackson Network} & \text{CM} & \text{SR} \\
\hline
L & 0.8707 & 0.8841 \\
L_q & 0.86 & 6.7419 \\
L & 6.73 & 7.633 \\
W & 0.4348 & 0.361 \\
\bar{W} & 0.5155 & 0.7199 \\
\rho_0 & 0.1293 & 0.1158 \\
\hline
\text{Other Approaches} & \text{CM} & \text{SR} \\
\hline
L & 0.8533 & 0.867 \\
L_q & 4.965 & 5.633 \\
L & 5.818 & 6.50 \\
\bar{W} & 0.387 & 0.542 \\
\bar{W} & 0.455 & 0.625 \\
\rho_0 & 0.1467 & 0.133 \\
\end{array}
\]

System is only an approximation.

Since the routing is state dependent, Jackson network assumes customers will return indefinitely.

(8) \Rightarrow (b)

(8) \Rightarrow (c)

\[W_s = 123.8 \text{ hrs} \]

\[64.8 \text{ min} \]

(9) \Rightarrow (a)

\[L = 14.36 \]

\[12.32 \text{ people} \]
Question IV: 30 points

The Copy Shop is open 5 days per week for copying materials that are brought to the shop. It has three identical copying machines that are run by employees of the shop. Only two operators are kept on duty to run the machines, so the third machine is a spare that is used only when one of the other machines breaks down. When a machine is being used, the time until it breaks down has an exponential distribution with a mean of 2 weeks. If one machine breaks down while the other two are operational, a service representative is called in to repair it, in which case the total time from the breakdown until the repair is completed has an exponential distribution with a mean of 0.2 week. However, if a second machine breaks down before the first one has been repaired, the third machine is shut off while the two operators work together to repair this second machine quickly, in which case its repair time has an exponential distribution with a mean of only 1/15 week.

If the service representative finishes repairing the first machine before the two operators complete the repair of the second, the operators go back to running the two operational machines while the representative finishes the second repair, in which case the remaining repair time has an exponential distribution with a mean of 0.2 week.

(a) Letting the state of the system be the number of machines not working construct the rate diagram for this queueing system.

(b) Use the balance equations to find the steady-state distribution of the number of machines not working.

(c) What is the expected number of operators available for copying?

\[ \lambda_0 = \frac{3}{2}, \quad \lambda_1 = 1 \]

\[ M_1 = 5, \quad M_2 = 20 = (15 + 5) \]

Using the rate diagram, the balance equations can be set up as follows:

\[ \begin{align*}
\text{State } 0: & \quad 5P_1 = \frac{3}{2}P_0 \\
\text{State } 1: & \quad \frac{3}{2}P_0 + 20P_2 = 6P_1 \\
\text{State } 2: & \quad P_1 = 20P_2
\end{align*} \]

Using the first and third balance equations along with \(P_0 + P_1 + P_2 = 1\), we can solve for the steady-state probabilities. Let's denote the probabilities as follows:

\[P_0, P_1, P_2\]

We can set up the system of equations:

\[ \begin{align*}
5P_1 &= \frac{3}{2}P_0 \\
\frac{3}{2}P_0 + 20P_2 &= 6P_1 \\
P_1 &= 20P_2
\end{align*} \]

From the third equation, we can express \(P_1\) in terms of \(P_2\):

\[P_1 = 20P_2\]

Substituting \(P_1\) into the first equation:

\[5(20P_2) = \frac{3}{2}P_0 \Rightarrow 100P_2 = \frac{3}{2}P_0 \Rightarrow \frac{3}{2}P_0 = 100P_2 \]

\[P_0 = \frac{200}{3}P_2 \]

Substituting \(P_1\) into the second equation:

\[\frac{3}{2}P_0 + 20P_2 = 6(20P_2) \Rightarrow \frac{3}{2}P_0 + 20P_2 = 120P_2 \]

\[\frac{3}{2}P_0 = 100P_2 \Rightarrow P_0 = \frac{200}{3}P_2 \]

We know that the sum of probabilities is 1:

\[P_0 + P_1 + P_2 = 1\]

Substituting the expressions for \(P_0\) and \(P_1\):

\[\frac{200}{3}P_2 + 20P_2 + P_2 = 1 \Rightarrow \frac{200}{3}P_2 + 21P_2 = 1 \Rightarrow \frac{361}{3}P_2 = 1 \]

\[P_2 = \frac{3}{361} \]

Finding \(P_0\) and \(P_1\):

\[P_0 = \frac{200}{3} \cdot \frac{3}{361} = \frac{200}{361} \]

\[P_1 = 20 \cdot \frac{3}{361} = \frac{200}{361} \]

The expected number of operators available for copying is:

\[E[\text{# of operators available}] = 2P_0 + 2P_1 + P_2 = 2 \cdot \frac{200}{361} + 2 \cdot \frac{200}{361} + \frac{3}{361} = \frac{523}{361} = 1.451 \]

Therefore, the expected number of operators available for copying is approximately 1.45.