Midterm Examination I

"Solutions"

Question #1: __________________/30
Question #2: __________________/30
Question #3: __________________/40

Grand Total $\sum$: _______________________________________

Examination Number #: ____________________________
Midterm Examination

Please Place your name of the BACK of the LAST PAGE of the examination!

There are three questions on this examination. The first question is a series of 5 (true-false) questions, each worth 6 points each for a total of 30 points. The other two questions are open-ended with point values as indicated. Please answer all questions and show all work on this examination paper. Please answer all questions and show all work on your examination paper. You may use the back of the sheets if necessary.

Question I: 30 points:

In the following true-false questions, answer all questions, there is no penalty for guessing. Please circle the appropriate response!

T F a) For an absorbing Markov Chain, the fundamental matrix is given as \( N = [I - Q]^{-1} \) and the components \( n_{ij} \) of \( N \) represent the expected number of times that the DTMC is in the transient state \( j \) given that it started in the transient state \( i \).

T F b) The expected recurrence time for state \( i \) in an ergodic DTMC is given by \( \mu_{ii} = \frac{1}{\pi_i} \).

T F c) For fixed \( i \) and \( j \), the probability distribution function for the first passage time to go from state \( i \) to state \( j \) is \( \sum_{n=1}^{\infty} f_{ij}^{(n)} \leq 1 \).

T F d) The following steady-state equations for a Continuous Time Markov Chain (CTMC) are called the "local balance" equations \( \pi_j q_j = \sum_{i \neq j} \pi_i q_{ij} \forall j \).

T F e) If the number of identical parts in a plant is 200, the chance of each part failing is 1%. The gain by replacement is $12,000 while the loss by not using it is $2000. The optimal number of spare parts to be kept in stock is about 6?
Question II: 30 points

Consider the DTMC that has the following (one-step) transition matrix:

\[
P = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
0 & \frac{4}{5} & 0 & \frac{1}{5} & 0 \\
1 & \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 \\
2 & 0 & \frac{1}{2} & 0 & \frac{1}{10} & \frac{2}{5} \\
3 & 0 & 0 & 0 & 1 & 0 \\
4 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0
\end{pmatrix}
\]

(20) a) Determine the classes of this DTMC and, for each class, determine whether it is recurrent or transient.

(10) b) For each of the classes identified in part (a), determine the period of the states in that class.

(a) Draw the transition diagram

Classes \{0, 1, 2, 4\} \{3\}

This is an Absorbing Markov Chain (AMC) \{0, 1, 2, 4\} is transient and \{3\} is recurrent

(b) For \{0, 1, 2, 4\} the period is 2

For class \{3\} the period is 1.

AMC
Question III: 40 points

A delicate precision instrument has a component that is subject to random failure. In fact, if the instrument is operating properly at any given moment in time, then with probability 0.1, it will fail in the next 10-minute period. If the component fails, it can be replaced by a new one, an operation that also takes 10 minutes. The present supplier of replacement components does not guarantee that all replacement components are in proper working condition.

The present quality standards are such that about 1% of the components supplied are defective. However, this can be discovered only after the defective component has been installed. If the component is defective, the instrument has to go through a new replacement operation. Assume that when a failure occurs, it is always at the end of a 10-minute period.

(10) a) Find the transition matrix associated with this process.

(10) b) Given that it was working properly initially, what is the unconditional probability that the instrument is not in proper working condition after 30 minutes?

(10) c) Please find the steady state probabilities of this DTMC. For what fraction of time is the instrument being repaired?

(10) d) Assume that each replacement component has a cost of 30 cents, and that the opportunity cost in terms of lost profit during the time the instrument is not working is $10.80 per hour. What is the average cost per 10-minute period?

\[ P = \begin{bmatrix}
0 & 1 \\
.10 & .90
\end{bmatrix} \]

\[ \pi = \begin{bmatrix}
.90 \\
.10
\end{bmatrix} \]

\[ \pi' = \begin{bmatrix}
.99 \\
.01
\end{bmatrix} \]

\[ \pi'' = \begin{bmatrix}
.90 \\
.10
\end{bmatrix} \]

\[ \pi''' = \begin{bmatrix}
.9091 \\
.09181
\end{bmatrix} \]

Thus, \( a_0 = .09181 \)
(c) \[ \bar{\pi} = \bar{\pi} P \Rightarrow \pi_0 = \pi_{0.01} + \pi_{1.10} \]
\[ \pi_0 + \pi_1 = 1 \]
Solving \[ \pi_0 - \pi_{0.01} - \pi_{1.10} = 0 \]
\[ \pi_0 + \pi_1 = 1 \]
(or)
\[ 0.99 \pi_0 - \pi_{1.10} = 0 \]
\[ \pi_0 + \pi_1 = 1 \]
(multiplying by 0.10)

\[ \downarrow \]
\[ 0.99 \pi_0 - 0.10 \pi_1 = 0.0 \]
\[ 0.10 \pi_0 + 1.10 \pi_1 = 0.10 \]
\[ 1.09 \pi_0 = 0.10 \]
\[ \Rightarrow \begin{cases} \pi_0 = 0.0917431 \\ \pi_1 = 0.9082568 \end{cases} \]

Fraction of instruments being replaced is \( \pi_0 = 0.0917 \)

(d) Long run average costs per year-minutes period
\[ \text{\$} (1.80 + 0.20) \pi_0 = \$ 0.192661 \]