Homework #4: Due in two weeks

1. (10 pts.) Consider the Discrete Time MDP with state space \{1, 2, 3\} and action space \{1, 2, 3\} with the following probability transition matrices:

\[
P(1) = \begin{pmatrix} .1 & .3 & .1 \\ 0 & .7 & .8 \\ 0 & .5 & .1 \end{pmatrix} \quad P(2) = \begin{pmatrix} .5 & .5 & 0 \\ 0 & .5 & .5 \\ 0 & .5 & .5 \end{pmatrix} \quad P(3) = \begin{pmatrix} .5 & .5 & 0 \\ .8 & 0 & .2 \\ 0 & .3 & .7 \end{pmatrix}
\]

and the following cost matrix:

\[
R = \begin{pmatrix} 30 & 10 & 40 \\ 30 & 10 & 5 \\ 20 & 50 & 10 \end{pmatrix}
\]

i) Set up the Linear Program formulation for the optimal MDP policy.

ii) Compute the optimal policy with the simplex method. You may use a computer to solve the LP but please include the print-out of the setup and solution.

2. (10 pts.) Consider a network with three arcs as shown in the following figure, Figure 1:

![3-node Network Diagram](image)

**Figure 1: 3-node Network**

Let \( X_i \) represent the length of arc \( i \). Suppose that \( X_i \sim \text{exp}(\lambda_i) \) are independent RVs. Compute the distribution of the shortest path from A to C.

3. (10 pts.) Consider again the network of Figure 1. Assume that \( X_i \sim \text{exp}(\lambda_i) \) are independent RVs. Compute the distribution of the longest path from node A to node C. (Assume that \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are distinct and \( \lambda_3 \neq \lambda_1 + \lambda_2 \).)

4. (10 pts.) Problem #5-21 (#5-21 in 7th edition)

5. (10 pts.) Problem #5-29 (#5-25 in 7th edition)

6. (10 pts.) Problem #5-61 (#5-58 in 7th edition)

7. (10 pts) A two-dimensional Poisson process is a process of randomly occurring points in a Euclidean plane such that

   i) the number of points in a region of area \( A \) is \( P(\lambda A) \)

   ii) the number of points in disjoint regions are independent of each other.

Consider an arbitrary point “a” in this plane. Let \( X_1 \) denote the distance from a to the nearest point. Compute the density of \( X_1 \). Hint: compute \( P\{X_1 > t\} \) first.
Homework 4

\[ E(c) = \sum_{i=1}^{3} \sum_{a=1}^{2} R_{ia} \pi_{ia} \]

\[ \pi_{11} + 10\pi_{12} + 40\pi_{13} + 30\pi_{21} + 10\pi_{22} + 5\pi_{23} + 20\pi_{31} + 50\pi_{32} + 10\pi_{33} \]

5.6. \[ \pi_{11} + \pi_{12} + \pi_{13} + \pi_{21} + \pi_{22} + \pi_{31} + \pi_{32} + \pi_{33} = 1 \]

Minimize 30p11 + 10p12 + 40p13 + 30p21 + 10p22 + 5p23 + 20p31 + 50p32 + 10p33

s.t.
\[ p11 + p12 + p13 + p21 + p22 + p23 + p31 + p32 + p33 = 1 \]
\[ p11 + p12 + p13 - 3p21 - 1p31 - 5p12 - 5p13 - 8p23 = 0 \]
\[ p21 + p22 + p23 - 1p31 - 8p31 - 5p22 - 5p23 - 3p32 - 3p33 = 0 \]
\[ p31 + p32 + p33 - 1p31 - 5p31 - 3p22 - 5p23 - 2p33 = 0 \]

LP optimum found at step 3

Objective function value

1) 8.469388

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<tr>
<th>VARIABLE</th>
<th>VALUE</th>
<th>REDUCED COST</th>
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<td>P33</td>
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Row slack or surplus | Dual prices
2) 0.00000 | 8.469388
3) 0.00000 | 2.040816
4) 0.00000 | 5.102041
5) 0.00000 | 0.000000

No. iterations = 3

Thus the state 2 \rightarrow 8
2 \rightarrow 3
3 \rightarrow 3

\[ z = 8.46939 \]
This can be done in a brute-force manner. We use the properties of exponentials.

Let $L$ be the length of the shortest path. If $X_3 \leq X_1$, $L = \min(X_3, X_1)$; else $L = \min(X_3, X_2) + \min(X_3 - X_1, X_2)$.

Given $X_3 \leq X_1$, $\min(X_1, X_3) \sim \exp(\lambda_1 + \lambda_2)$ and given $X_3 > X_1$, $X_3 - X_1 \sim \exp(\lambda_3)$ and $\min(X_3 - X_1, X_2) \sim \exp(\lambda_2 + \lambda_3)$.

$$P(L \leq x) = P(L \leq x | X_3 \leq X_1) P(X_3 \leq X_1) + P(L \leq x | X_3 > X_1) P(X_3 > X_1)$$

$$= P(\min(X_3, X_1) \leq x | X_3 \leq X_1) P(X_3 \leq X_1) +$$

$$P(\min(X_1, X_3) + \min(X_3 - X_1, X_2) \leq x | X_3 > X_1) P(X_3 > X_1)$$

$$= P(\exp(\lambda_1 + \lambda_3) \leq x) P(X_3 \leq X_1) + P(\exp(\lambda_1 + \lambda_3)$$

$$+ \exp(\lambda_2 + \lambda_3) \leq x) P(X_3 > X_1)$$

$$= \frac{\lambda_3}{\lambda_1 + \lambda_3} \left(1 - e^{-(\lambda_1 + \lambda_2)x}\right)$$

$$+ \frac{\lambda_1}{\lambda_1 + \lambda_3} \left\{\frac{\lambda_2 + \lambda_3}{\lambda_2 - \lambda_1} \left(1 - e^{-(\lambda_1 + \lambda_3)x}\right) + \frac{\lambda_1 + \lambda_3}{\lambda_1 - \lambda_2} \left(1 - e^{-(\lambda_2 + \lambda_3)x}\right)\right\}$$

$$= \frac{\lambda_2}{\lambda_2 - \lambda_1} \left(1 - e^{-(\lambda_1 + \lambda_3)x}\right) + \frac{\lambda_1}{\lambda_1 - \lambda_2} \left(1 - e^{-(\lambda_2 + \lambda_3)x}\right).$$

Let $L$ be the length of the longest path.

$$P(L \leq x) = P(\max(X_1, X_2, X_3) \leq x)$$

$$= P(X_1 + X_2 \leq x, X_3 \leq x)$$

$$= P(X_1 + X_2 \leq x) P(X_3 \leq x)$$

$$= \left(1 - \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{\lambda_1 x} - \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_2 x}\right) (1 - e^{\lambda_3 x}).$$
\[ \text{E}[\text{time}] = \text{E}[\text{time waiting at 1}] + \frac{1}{M_1} + \text{E}[\text{time waiting at 2}] + \frac{1}{M_2} \]

Now
\[ \text{E}[\text{time waiting at 1}] = \frac{1}{M_1} \]
\[ \text{E}[\text{time waiting at 2}] = \left(\frac{1}{M_2}\right) \frac{M_1}{M_1 + M_2} \]

The latter equation follows by conditioning on whether or not the customer waits for server 2. Therefore
\[ \text{E}[\text{time}] = \frac{2}{M_1} + \left(\frac{1}{M_2}\right) \left[1 + \frac{M_1}{M_1 + M_2}\right] \]

\[ \frac{\lambda}{\lambda + M_A} \]

(b) \[ \frac{\lambda + M_A}{\lambda + M_A + M_B} \cdot \frac{\lambda}{\lambda + M_B} \]

\[ \text{Poisson with mean } cG(t) \]
\[ \text{Poisson with mean } c[1 - G(t)] \]
\[ \text{Independent} \]

Note that \( X > t \) if and only if there are no points in a circle of radius \( t \) centered at \( a \), i.e., in a region of area \( \pi t^2 \). Since the number of points in this region is \( P(\lambda \pi t^2) \), we can see that the desired probability is \( \exp\{-\lambda \pi t^2\} \).